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On critical circle homeomorphisms

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—*Dedicated to the memory of R. Mañé*

Abstract. We prove that an analytic circle homeomorphism without periodic orbits is conjugated to the linear rotation by a quasi-symmetric map if and only if its rotation number is of constant type. Next, we consider automorphisms of quasi-conformal Jordan curves, without periodic orbits and holomorphic in a neighborhood. We prove a "Denjoy theorem" that such maps are conjugated to a rotation on the circle.

Keywords: Cross-ratio inequality, quasi-symmetric conjugacy, Denjoy theorem.

1. Circle Maps

1.1. Introduction

Apparently due to the great complexity of problems involved, there is a tendency to divide dynamical systems into ever smaller sub-fields, each developing in its own right. Only exceptional mathematicians are able to overcome this tendency. Ricardo Mañé was this type of researcher. He left a deep mark on ergodic theory, general theory of diffeomorphisms on surfaces, hamiltonian dynamics, and the systems in one real or complex dimension. To him, dynamical systems was one field.

This paper is not directly connected to Mañé's research. Without trying to approach his scope of vision, we show another example of a close connection between the theory in one real and complex dimension. Both sub-fields are quite different in the tools they use: strong reliance on the ordering of points in real dynamics and methods of complex function theory in holomorphic dynamics. Interactions between these theories have repeatedly been shown to lead to new deep results. An

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example of this occurred in 1987 when Michel Herman proved a theorem about boundaries of Siegel disks with rotation numbers of constant type. His solution was based on quasiconformal surgery, see [**4**], and a theorem about the quasi-symmetric conjugacy between an analytic homeomorphism of the circle and a linear rotation, see [**3**].

In the first section we provide a proof of Herman's theorem about circle mappings. Our proof is somewhat different from the original one and based on the method of [**2**]. We introduce a new technical concept of a cross-ratio module as the "minimal" tool which makes the theory work. Another purpose of Section 1 is to simply fill out a gap in the literature by publishing proofs of results which have been referenced several times.

The second section contains a generalization of the theorem by Yoccoz concerning the absence of Denjoy counterexamples for analytic maps. We extend the result to holomorphic maps that preserve an arbitrary quasiconformal Jordan curve. This part of our work provides a good example of the interplay between "real" arguments based on the ordering of points and complex function-theoretic tools.

Quasi-symmetric homeomorphisms of the line.

Definition 1.1. *A homeomorphism* $h : \mathbb{R} \to \mathbb{R}$ *is called* Q-quasi-symmetric *if and only if for every real* x *and* $\delta \neq 0$

$$\frac{|h(x+\delta) - h(x)|}{|h(x) - h(x-\delta)|} \leq Q.$$

Rotation numbers. If $f\colon \mathbb{R} \to \mathbb{R}$ is the lifting of a degree 1 homeomorphism of the circle, i.e. $f(x+1) = f(x) + 1$ and f is increasing, then the following limit:

$$\rho(f) := \lim_{n\to\infty} \frac{f^n(x)}{n}$$

exists for every x and is independent of x. It is called the *rotation number* of f. A lifting can always be chosen so that $\rho(f) \in [0,1)$, so we assume that in the sequel. An irrational rotation number can be uniquely represented in the form of an infinite continued fraction

$$\rho(f) = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

with $a_1, \ldots$ all positive integers. We say that the rotation number is of *constant type* if and only if the numbers a_i are all bounded by a constant.

The statement of M. Herman's theorem.

Theorem 1.1. *Let f be a real-analytic homeomorphism of the real line onto itself, a lifting of a degree 1 circle homeomorphism. Suppose that the derivative of f vanishes at least at one point. Let $\rho(f) \notin \mathbb{Q}$ be the rotation number of f.*

Then there is a homeomorphism $H: \mathbb{R} \to \mathbb{R}$, also a lifting of a degree 1 circle map, so that the functional equation

$$H(x + \rho(f)) = f(H(x))$$

is satisfied. Moreover, H is quasi-symmetric if and only if $\rho(f)$ is of constant type.

The existence of H which solves the given functional equation had followed from an earlier theorem of Yoccoz (see [**11**]). The proof of the whole theorem appeared in the manuscript [**3**].

Sketch of the paper. We will re-state Theorem 1.1 in a more general form which does not assume any smoothness and instead is fully based on an estimate of the cross-ratio distortion. We will proceed to prove this theorem. Since real-analytic maps satisfy the cross-ratio condition, Herman's theorem follows.

1.2. Bounded Geometry

Cross-ratios. Choose four points on the real line so that either

$$a \le b < c \le d$$

or all inequalities are reversed. Then define their *cross-ratio* by

$$\mathbf{Cr}(a, b, c, d) := \frac{|a-b| \cdot |c-d|}{|a-c| \cdot |d-b|}.$$

It turns out that we can consider a much more general class of functions of the quadruples of points, which will still be sufficient for our proofs.

Definition 1.2. *A* cross-ratio module *is a function* χ *from all quadruples of points of the real line which satisfy* $a \leq b < c \leq d$, *or* $a \geq b > c \geq d$, *with values in* $[0, \infty)$ *provided that:*

- *there is a constant* C *so that if* $\mathbf{Cr}(a, b, c, d) \geq \frac{1}{4}$, *then* $\chi(a, b, c, d) \geq C$,
- *for every* $\epsilon > 0$ *there is a* $\delta > 0$ *so that if* $\mathbf{Cr}(a, b, c, d) < \delta$, *then* $\chi(a, b, c, d) < \epsilon$.

The cross-ratio inequality. Suppose that n quadruples of points $a_i < b_i < c_i < d_i$ are chosen on the real line. We will say that they are in an *allowable configuration* if the intervals (a_i, d_i) are pairwise disjoint modulo 1 and $d_i - a_i < 1$.

Definition 1.3. *Let* $g: \mathbb{R} \to \mathbb{R}$ *be strictly increasing and suppose that* $g(x) - x$ *is* 1*-periodic. Let* χ *be a cross-ratio module. We say that* g *satisfies the* cross-ratio inequality with respect to χ *(CRI-*χ *in short) with bound* Q *if and only if for any choice of quadruples of points* (a_i, b_i, c_i, d_i), $i = 1, \ldots, n$ *in an allowable configuration, the estimate*

$$\prod_{i=1}^{n} \frac{\chi(g(a_i), g(b_i), g(c_i), g(d_i))}{\chi(a_i, b_i, c_i, d_i)} \leq Q$$

holds true.

A technical remark. In earlier works, [**9**], the cross-ratio inequality was stated in a seemingly stronger form. Instead of the allowable configurations all quadruples were allowed. Then, one defined an intersection number for a configuration equal to the supremum over points $x \in \mathbb{R}$ of k so that x is contained modulo 1 in k intervals (a_i, d_i) from the configuration. For example, the configuration is allowable in the sense defined earlier if and only if its intersection number is 1. In these earlier works the constant Q in the cross-ratio inequality depended on the intersection number.

A graph-theoretical argument shows that our cross-ratio inequality implies this older version.

Lemma 1.1. *Suppose that g satisfies the cross-ratio inequality with respect to some χ with bound Q. Let (a_i, b_i, c_i, d_i) be any configuration of quadruples of points on the real line with intersection number k. Then*

$$\prod_{i=1}^{n} \frac{\chi(g(a_i), g(b_i), g(c_i), g(d_i))}{\chi(a_i, b_i, c_i, d_i)} \leq Q^{2k}$$

Proof. The lemma will be proven if we can divide the set of intervals (a_i, d_i) into $2k$ classes so that the intervals in each class are disjoint on the circle and thus form an allowable configuration. Pick the intervals (a_i, d_i) that contain an integer and make each a one-element class. This gives us no more than k classes. The remaining intervals can be moved by integer translations onto the interval $(0, 1)$. Let $\mathcal{I}$ be the set of the moved intervals, with those containing integers excluded. The intervals from $\mathcal{I}$ have the intersection number k and we will prove that they can be divided into no more than k classes of disjoint intervals.

Consider the graph G having the elements of $\mathcal{I}$ as vertices. An edge joins two intervals if and only if they intersect. Such a graph is called an *interval graph* (see [**1**], page 13). It is known that every interval graph is *perfect* (ibidem, Theorem 4.11 on page 95 and Problem 2 on page 100), i.e. its vertices can be separated into κ classes ("colors") so that no two vertices of the same color are adjacent, where κ is the number of vertices of the largest complete subgraph of G. Recall here that a complete graph, also known as a clique, is a graph with all possible connections. "Largest" in the previous sentence means "having the greatest number of vertices". A complete subgraph of G corresponds to a set of intervals with non-empty pairwise intersections. They must have one point in common, hence $\kappa = k$. The coloring gives us exactly the needed division of the set of intervals into k classes so that no two intervals in the same class intersect. □

Checking the cross-ratio inequality. In [**9**], it is proven that if g is real-analytic, then g satisfies the cross-ratio inequality with respect to the classical cross-ratio with some bound. Hence, the mapping f from the hypothesis of Theorem 1.1 satisfies the CRI-**Cr**.

In the situation of finitely many critical points of g (counted modulo 1), the method of checking the CRI-**Cr** consists of three steps. First, one checks that if (a_i, d_i) is in a small neighborhood of a critical point of the mapping, but does not contain the critical point, the cross-ratio is decreased under the action of g. Secondly, if (a_i, d_i) contains a critical point, then the cross-ratio shows a bounded growth. This is close to assuming that g itself is quasi-symmetric. Third, the product of changes on all other quadruples is bounded. This last step ignores the critical points and so methods developed for the study of diffeomorphisms such as bounded variation or the Zygmund property apply. More information about checking the cross-ratio inequality and estimating the bound can be found in [**5**], [**8**], [**6**] and [**10**]. In the final section of this paper we will work with a non-classical cross-ratio module to show that the cross-ratio inequality is satisfied.

1.3. Quasi-symmetry of orbits

Facts from arithmetic. If $a_1, \cdots, a_n, \cdots$ are coefficients of the continued fraction representation of $\rho(f)$, then let p_n/q_n, $n \geq 1$, denote the n-th *convergent*, i.e. the rational number, written in the simplest terms, that is obtained after dropping the part of the continued fraction which comes after a_n. For example,

$$\frac{p_2}{q_2} = \frac{1}{a_1 + \frac{1}{a_2}}.$$

Dynamically, the numbers q_n are the closest return times for the rotation by $\rho(f)$, i.e.

$$|\rho(f)q_n - p_n| \leq |\rho(f)k - m|$$

for any two positive integers k, m so that $k < q_{n+1}$ with the equality only when $k = q_n$ and $m = p_n$.

The following fact describes the ordering of orbits under circle homeomorphisms.

Fact 1.1. *Let $g:\mathbb{R}\to\mathbb{R}$ be increasing and satisfy $g(x+1)=g(x)+1$ for every $x\in\mathbb{R}$. Choose Q positive so that $\rho(g)=P/Q$ in simplest terms, or make $Q=\infty$ if $\rho(g)$ is irrational. Then for every $x\in\mathbb{R}$ and every pair of integers p,q so that $|q|<Q$, we have*

$$(f^q(x)-x-p)(q\rho(g)-p)>0.$$

Proof. This is an easy exercise. □

A statement about quasi-symmetry. Let us state a lemma.

Lemma 1.2. *Suppose that $f:\mathbb{R}\to R$ is a lifting of a degree 1 circle homeomorphism with rotation number which is irrational. Assume also that f satisfies the cross-ratio inequality with respect to a cross-ratio module χ with bound M. Let p_n/q_n be a convergent of $\rho(f)$.*

Then, for every M, there is a $K\geq 1$, also depending on χ, so that for every $x\in\mathbb{R}$

$$K^{-1}|f^{q_n}(x)-p_n-x|\leq|f^{-q_n}(x)+p_n-x|\leq K|f^{q_n}(x)-p_n-x|.$$

Proof. Without loss of generality assume that $p_n/q_n<\rho(f)$. Otherwise, we could consider $F(x)=-f(-x)+1$. We see that $\rho(F)=1-\rho(f)$, and so the n-th convergent of $\rho(F)$ would be $1-\frac{p_n}{q_n}$. We could then do the argument for F instead of f and for $-x$ instead of x to obtain the same estimate.

Adopt notations $P/Q:=p_n/q_n$ and P'/Q' for the next fraction larger than P/Q, with $Q'\leq Q$. Notice that $P'/Q'>\rho(f)$. Otherwise, we would have

$$0<\rho(f)-\frac{P'}{Q'}<\rho(f)-\frac{P}{Q}$$

which would imply

$$|Q'\rho(f)-P'|<|Q\rho(f)-P|$$

contrary to the assumption that Q is a closest return time.

Let us fix a point $x\in\mathbb{R}$. From Fact 1.1 we see that $f^Q(x)-P>x$ and $f^{Q'}(x)-P'<x$. Hence, $f^{-Q'}(x)+P'>x$. Choose n so that $f^{-Q'}(x)+P'$ lies between $f^{(n-1)Q}(x)-(n-1)P$ and $f^{nQ}(x)-nP$. Observe that $n\geq 2$ since $n=1$ would imply by Fact 1.1 that $0<-Q'\rho(f)+P'<Q\rho(f)-P$

in contradiction to Q being a closest return. For $n = 2$, the ordering of points is shown on Figure 1.

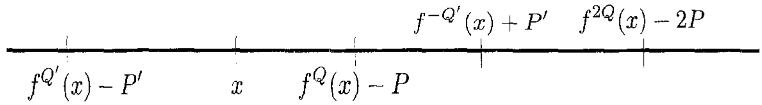

Figure 1 – The ordering of points in the proof of Lemma 1.2.

Now take $F_s := f - s$ where s is a non-negative number. As s grows, point of the forward orbit of x move to the left and those in the backward orbit proceed to the right. Hence there is a unique value of s for which $F_s^{nQ}(x) - nP = F_s^{-Q'}(x) + P'$. Let fix s at this value, and write F for F_s in order to keep the notation simple. Clearly, the orbit of x by F is periodic with rotation number

$$\rho' = \frac{nP + P'}{nQ + Q'}.$$

Note the following ordering of points

$$0 < F^{Q}(x) - P < f^{Q}(x) - P < F^{2Q}(x) - 2P. \tag{1}$$

Of these inequalities, the last one is not obvious. If it didn't hold, we would have

$$\begin{aligned} F^{nQ}(x) - nP &= F^{(n-2)Q}(F^{2Q}(x) - 2P) - (n-2)P \\ &\leq F^{(n-2)Q}(f^{Q}(x) - P) - (n-2)P \\ &\leq f^{(n-2)Q}(f^{Q}(x) - P) - (n-2)P \\ &= f^{(n-1)Q}(x) - (n-1)P. \end{aligned}$$

So,

$$F^{nQ}(x) - nP \leq f^{(n-1)Q}(x) - (n-1)P < f^{-Q'}(x) + P' < F^{-Q'}(x) + P'$$

which is a contradiction. By the same reasoning, we get

$$0 > F^{-Q}(x) + P > f^{-Q}(x) + P > F^{-2Q}(x) + 2P. \tag{2}$$

The orbit of $e^{2\pi i x}$ by the projection of F consists of $nQ + Q'$ points. Notice that the projections of x and $F^{Q}(x)$ are consecutive. Otherwise, for some $0 < q < nQ + Q'$ we would have

$$x < F^q(x) - p < F^Q(x) - P$$

or, by Fact 1.1

$$0 < q\rho' - p < Q\rho' - P. \tag{3}$$

Clearly, ρ' exceeds both p/q and P/Q.

We claim that $p/q > P/Q$. First, note that $q > Q$. Since $\rho' < \rho(f)$ and by Fact 1.1, we have $0 < q\rho' - p < q\rho(f) - p$, and since P/Q is a closest return time for the rotation by ρ, $0 < Q\rho(f) - P < q\rho(f) - p$. But since $\rho(f) > \rho'$ this means that

$$Q\rho' - P < q\rho' - p,$$

a contradiction with inequality (3). Another thing which follows from the same inequality is that

$$\rho' < \frac{p - P}{q - Q} = u,$$

hence p/q is between u and P/Q, thus $p/q > P/Q$ as claimed.

However, $p/q \in (P/Q, \rho')$ is impossible since P/Q and ρ' are *Farey neighbors.* The term Farey neighbors refers to two fractions, written is the simplest terms, so that the denominator of any fraction between them exceeds the denominators of both Farey neighbors. From the way P/Q and P'/Q' were chosen, they were obviously Farey neighbors. A general principle, see [**9**], Section 3, is that if r/s and r'/s' are Farey neighbors, then so are r/s and $(r + r')/(s + s')$. Applying this principle n times, we see that P/Q and ρ' are indeed Farey neighbors. So, the projections of x and $F^Q(x)$ are consecutive points of the orbit of x.

Let $\mathcal{I}$ denote the collection of arcs on the circle with endpoints at two consecutive points of the orbits of x by F.

Claim . *Let χ be a cross-ratio module. If F satisfies CRI-χ with bound M and if $I_1, I_2 \in \mathcal{I}$ are adjacent, then $|I_1|/|I_2| \geq M'$ where $M' > 0$ depends only on M and χ.*

Before proving the claim, note that it immediately implies Lemma

1.2. Indeed, it means that all four intervals delimited by points

$$F^{-2Q}(x) + 2P \;, F^{-Q}(x) + P \;, x \;, F^{Q}(x) - P \quad \text{and} \quad F^{2Q}(x) - 2P$$

have lengths comparable with the factor of $(M')^{-3}$, hence controlled by M. In view of inequalities (1) and (2), Lemma 1.2 follows with $K := 2(M')^{-3}$.

The claim is essentially Lemma 8 of [**9**] and the proof goes as follows. Choose $I_3 \in \mathcal{I}$ adjacent to I_2 on the opposite side. Lift I_1, I_2, I_3 to the line to get intervals (a,b), (b,c) and (c,d), respectively. Observe that

$$\mathbf{Cr}(a,b,c,d) < \frac{|I_1|}{|I_2|} \tag{4}$$

Then choose the smallest ℓ so that f^ℓ maps I_2 to the shortest arc in $\mathcal{I}$. The configuration (a,b,c,d), ..., $(f^{\ell-1}(a), \ldots, f^{\ell-1}(d))$ has intersection number at most 3. From Definition 1.2, $\chi(f^\ell(a), f^\ell(b), f^\ell(c), f^\ell(d)) \geq C_1$. So, by Lemma 1.1,

$$C \leq \chi(f^\ell(a), \cdots, f^\ell(d)) \leq M^6 \chi(a,b,c,d).$$

So, $\chi(a,b,c,d) \geq CM^{-6}$. From Definition 1.2, $\mathbf{Cr}(a,b,c,d) \geq \delta(CM^{-6})$ and invoking estimate (4) ends the proof of the claim, and thus of Lemma 1.2. □

1.4. Bounded ratio of closest returns.

We will present a second property of orbits. Unlike the first one, proved in Lemma 1.2, this will require f to have critical points. Because of our tendency to avoid assuming any smoothness, we state a weaker property which gets used in the argument. Also, in this argument we assume that the cross-ratio inequality is satisfied with respect to the standard cross-ratio **Cr**. It would be possible to weaken this assumption to allow some cross-ratio module with extra properties. Since we have no good example when such a generalization would lead to new results, this seemed superfluous.

Definition 1.4. *Let $g\colon (a,b) \to \mathbb{R}$ be strictly increasing. We say that c is a* pseudo-critical point *of g if for every $\Gamma, \Delta > 0$ there is $\delta > 0$ so that* *for every* $x, y \neq c$ *if*

$$|x-c|/\delta < |y-c| < \Delta ,$$

then

$$\frac{|g(y)-g(c)|}{|g(x)-g(c)|} \geq \Gamma \frac{|y-c|}{|x-c|} .$$

As an easy exercise, we leave it to the reader to show that if g is C^1 in a neighborhood of c, then c is pseudo-critical if and only if $g'(c) = 0$.

Let us now state the result.

Lemma 1.3. *Suppose that* $f\colon \mathbb{R} \to \mathbb{R}$ *is a lifting of a degree* 1 *circle homeomorphism with the rotation number irrational. Let* p_{n+1}/q_{n+1} *and* p_n/q_n *be consecutive convergents of* $\rho(f)$. *Assume also that* f *satisfies the cross-ratio inequality with respect to* **Cr**.

Then, if c *is a pseudo-critical point of* f, *there is a constant* $K > 0$ *so that*

$$|f^{q_{n+1}}(c) - p_{n+1} - c| \geq K|f^{q_n}(c) - p_n - c| .$$

Proof. As in the proof of Lemma 1.2, we assume without loss of generality that $p_n/q_n < \rho(f)$, it follows then that $p_{n+1}/q_{n+1} > \rho(f)$. Start by observing a general fact the for every $x \in \mathbb{R}$, the projections to the circle of intervals $(x, x^{q_n}), \ldots , (f^{q_{n+1}-1}(x), f^{q_{n+1}+q_n-1}(x))$ are disjoint. By Fact 1.1 it will be enough to check this fact for the rotation by $\rho(f)$ instead of f, and then it is clear from q_n being a closest return.

Now choose points $d_0 = f^{q_{n+1}}(c) - p_{n+1}$, $c_0 = c$, $b_0 = f^{q_n}(c) - p_n$ and $a_0 = f^{2q_n}(c) - 2p_n$. Setting $*_i = f^i(*)$ where $*$ stands for a, b, c, d and $i = 1, \ldots , q_{n+1}$ we get a configuration of quadruples of points. The ordering of points is shown on Figure 2.

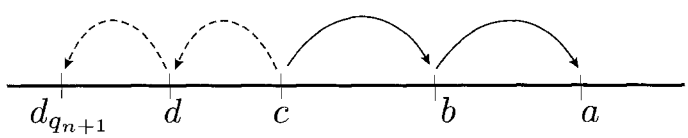

Figure 2 – Solid arrows denote the action of $f^{q_n} - p_n$; the dashed ones show $f^{q_{n+1}} - p_{n+1}$.

By the observation of the preceding paragraph, the intersection number of this configuration is no more than 3. Hence, by Lemma 1.1,

$$\mathbf{Cr}(d_{q_{n+1}}, c_{q_{n+1}}, b_{q_{n+1}} a_{q_{n+1}}) \leq Q^6 \ \mathbf{Cr}(d_1, c_1, b_1, a_1) \ . \tag{5}$$

In the forthcoming estimates we will use K_i for positive numbers independent of n. Let us estimate $\mathbf{Cr}(d_{q_{n+1}}, c_{q_{n+1}}, b_{q_{n+1}} a_{q_{n+1}})$ from below.

$$\mathbf{Cr}(d_{q_{n+1}}, c_{q_{n+1}}, b_{q_{n+1}} a_{q_{n+1}}) = \frac{|d_{q_{n+1}} - c_{q_{n+1}}|}{|d_{q_{n+1}} - b_{q_{n+1}}|} \frac{|b_{q_{n+1}} - a_{q_{n+1}}|}{|c_{q_{n+1}} - a_{q_{n+1}}|} \ .$$

Since $a_{q_{n+1}}$ and $c_{q_{n+1}}$ are the image and preimage of $b_{q_{n+1}}$, respectively, the second fraction is bounded from below by Lemma 1.2. Next, the interval $(d_{q_{n+1}}, b_{q_{n+1}})$ is contained in $(f^{-2q_n}(c) + 2p_n, b)$. So,

$$\mathbf{Cr}(d_{q_{n+1}}, c_{q_{n+1}}, b_{q_{n+1}} a_{q_{n+1}}) \geq K_1 \frac{|d_{q_{n+1}} - c_{q_{n+1}}|}{|f^{-2q_n}(c) + 2p_n - b|} =$$

$$= K_1 \frac{|d-c|}{|d-b|} \frac{|d_{q_{n+1}} - c_{q_{n+1}}|}{|d-c|} \frac{|d-b|}{|f^{-2q_n}(c) + 2p_n - b|} \ .$$

In the last estimate, the second fraction is bounded from below by a positive number from Lemma 1.2 since $d_{q_{n+1}}$ and c are the image and preimage, respectively, of d by $f^{q_{n+1}}$. The third fraction is bounded as follows:

$$\frac{|d-b|}{|f^{-2q_n}(c) + 2p_n - b|} \geq \frac{|c-b|}{|f^{-2q_n}(c) + 2p_n - b|} \ .$$

Since $c = f^{-q_n}(b)$, this is bounded from below by a positive constant from Lemma 1.2.

Hence,

$$\mathbf{Cr}(d_{q_{n+1}}, c_{q_{n+1}}, b_{q_{n+1}} a_{q_{n+1}}) \geq K_2 \frac{|d-c|}{|d-b|} \ . \tag{6}$$

Next, estimate

$$\mathbf{Cr}(d_1, c_1, b_1, a_1) < \frac{|d_1 - c_1|}{|d_1 - b_1|} < \frac{|d_1 - c_1|}{|c_1 - b_1|} \ .$$

From this and estimate (5), we get

$$\mathbf{Cr}(d_{q_{n+1}}, c_{q_{n+1}}, b_{q_{n+1}} a_{q_{n+1}}) < Q^6 \frac{|d_1 - c_1|}{|c_1 - b_1|} \ ,$$

while if take into account estimate (6)

$$\frac{|d-c|}{|d-b|} < K_3 \frac{|d_1 - c_1|}{|c_1 - b_1|} . \tag{7}$$

Now we use the fact that c is a pseudo-critical point. In Definition 1.4, set $\Delta = 2$ and $\Gamma = 2K_3$ to obtain the δ, which we can additionally make less or equal to 1. If $|d-c| < \delta|c-b|$, then

$$\frac{|d_1 - c_1|}{|c_1 - b_1|} \le \frac{1}{2K_3} \frac{|d-c|}{|c-b|} \le K_3^{-1} \frac{|d-c|}{|d-b|}$$

in clear contradiction to estimate (7). Hence $|d-c| \ge \delta|b-c|$ but this is just the claim of Lemma 1.3. □

1.5. Conjugacy Theorems

Based on Lemmas 1.2 and 1.3, we prove results about the quasi-symmetric conjugacy of a circle homeomorphism to the linear rotation.

A condition for QS conjugacy

Proposition 1. *Let $f: \mathbb{R} \to \mathbb{R}$ be a lifting of a degree 1 circle homeomorphism with an irrational rotation number $\rho(f)$. Suppose that f satisfies the cross-ratio inequality with bound Q with respect to some cross-ratio module χ.*

Then, there is $H: \mathbb{R} \to \mathbb{R}$, also a lift of a degree 1 circle homeomorphism, which conjugates f to the translation by $\rho(f)$. Furthermore, H is quasi-symmetric if $\rho(f)$ is of bounded type. If the continued fraction coefficients are all bounded by N, then H is K-quasi-symmetric where K depends only on Q, χ and N. If f has at least one pseudo-critical point and $\chi = \mathbf{Cr}$, then H is quasisymmetric if and only if $\rho(f)$ is of constant type.

Before proving Proposition 1, observe that it directly implies Theorem 1.1. The proof will naturally split into three steps: the construction of H, the proof that it is quasi-symmetric in the bounded type situation, and the proof that it is not quasi-symmetric otherwise.

The existence of a conjugacy. We will construct H to solve the functional equation

$$f(H(x)) = H(x + \rho(f)) .$$

The method is very natural. Make $H(0) = 0$. Then necessarily $H(m\rho(f) + n) = f^m(0) + n$ for any pair of integers m, n. This defines H on a dense subset of $\mathbb{R}$. By Fact 1.1, H is increasing. To prove that H extends to a homeomorphism of the line, it is necessary and sufficient that the closure of the set $S := \{f^m(0) + n : m, n \in \mathbb{Z}\}$ be the whole real line. If not, choose a point x on its boundary. Points $f^{q_{2n}}(x) - p_{2n}$ approach x from the right unless x has a right-sided neighborhood disjoint from S. Similarly, points $f^{-q_{2n}}(x) + p_{2n}$ approach x from the left unless x has a left-sided neighborhood disjoint from S. In view of Lemma 1.2, convergence cannot happen on only one side. Hence, all boundary points of $\overline{S}$ are isolated in $\overline{S}$. In particular, there are only finitely many of them modulo 1 and since map preserves S as well as ∂S, it will eventually map one of them into itself creating a periodic point on the circle, contrary to the assumption that $\rho(f)$ is irrational.

So the existence of H has been proved. Moreover, we showed that H is unique once the image of one point has been fixed, so all other conjugacies will be of the form $H(x+\lambda)$ where λ is an arbitrary constant. Hence to study their quasi-symmetry it does not matter which one is picked out.

The quasi-symmetry of H. Let us now assume that $\rho(f)$ is of constant type with continued fraction coefficients bounded by N. One has to recall that

$$a_n|q_n\rho(f) - p_n| \leq |q_{n-1}\rho(f) - p_{n-1}| \leq (a_n + 1)|q_n\rho(f) - p_n|$$

for all $n \geq 1$ with the convention $p_0/q_0 = 1/0$. By Fact 1.1, a consequence of this is that

$$|f^{\epsilon(a_n+1)q_n}(y) - \epsilon(a_n + 1)p_n - y| > |f^{-\epsilon q_{n-1}}(y) + \epsilon p_{n-1}| \qquad (8)$$

for all $n \geq 1$, $y \in \mathbb{R}$ and $\epsilon = -1, 1$.

Take $x \in \mathbb{R}$ and $1 \geq t > 0$. Choose $n \geq 0$ so that

$$|q_{n+1}\rho(f) - p_{n+1}| < t \leq |q_n\rho(f) - p_n| .$$

Then

$$H(x+t) \in \big(f^{-\epsilon q_{n+1}}(H(x)) + \epsilon p_{n+1}, f^{\epsilon q_n}(H(x)) - \epsilon p_n\big]$$

and likewise

$$H(x-t) \in \big[f^{-\epsilon q_n}(H(x)) + \epsilon p_n, f^{\epsilon q_{n+1}}(H(x)) - \epsilon p_{n+1}\big)$$

where $\epsilon = (-1)^n$. By relation (8),

$$\frac{|f^{-\epsilon q_{n+1}}(H(x)) + \epsilon p_{n+1} - H(x)|}{|f^{\epsilon(N+1)q_{n+1}}(H(x)) - (N+1)\epsilon p_{n+1} - H(x)|} \leq$$

$$\leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq \frac{|f^{-\epsilon(N+1)q_{n+1}}(H(x)) + (N+1)\epsilon p_{n+1} - H(x)|}{|f^{\epsilon q_{n+1}}(H(x)) - \epsilon p_{n+1} - H(x)|}$$

for $n \geq 0$. By Lemma 1.2 any adjacent two of the intervals with endpoints

$$f^{kq_{n+1}}(H(x)) - kp_{n+1} \quad \text{and} \quad f^{(k+1)q_{n+1}}(H(x)) - (k+1)p_{n+1}$$

for $k \in \mathbb{Z}$ are comparable within the multiplicative factor K where K depends only on the cross-ratio bound and χ. Hence

$$K^{-N-1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq K^{N+1}$$

for $0 < t \leq 1$.

It remains to consider the case of $t > 1$. If the integer part of t in m, clearly

$$\frac{n}{n+1} \leq \frac{|H(x+t) - H(x)|}{|H(x) - H(x-t)|} \leq \frac{n+1}{n}$$

because $H(x+n) = H(x) + n$. We have proved that H is $\max(2, K^{N+1})$-quasi-symmetric.

The necessity of constant type. We start with the following fact about quasi-symmetric homeomorphisms:

Fact 1.2. *A homeomorphism $h: \mathbb{R} \to \mathbb{R}$ is quasi-symmetric if and only if for every $\gamma > 0$ there is a $\gamma' > 0$ so that for any triple of distinct points x, y, y', with x placed between y and y', if*

$$\frac{|x-y|}{|x-y'|} \geq \gamma\,,$$

then

$$\frac{|h(x) - h(y)|}{|h(x) - h(y')|} \geq \gamma' .$$

The proof is an easy exercise.

If we now assume that $\rho(f)$ is not of constant type, we can find a sequence n_j so that $a_{n_j} \to \infty$. This means that for every x

$$\frac{|(q_{n_j}\rho(f) - p_{n_j} + x) - x|}{|(q_{n_j-1}\rho(f) - p_{n_j-1} + x) - x|} \to 0 .$$

On the other hand, if we choose x equal to the preimage of a pseudo-critical point of f, Lemma 1.3 implies that

$$\frac{|H(q_{n_j}\rho(f) - p_{n_j} + x) - H(x)|}{|H(q_{n_j}\rho(f) - p_{n_j} + x) - H(x)|}$$

remains bounded from below by a positive constant independent of j. In view of Fact 1.2 this means that H^{-1} is not quasi-symmetric, hence neither is H.

This concludes the proof of Proposition 1.

2. A Denjoy Theorem

We will describe a situation in which the cross-ratio inequality holds in a non-classical situation. As a result, we get a Denjoy theorem for analytic maps on quasiconformal Jordan curves.

Theorem 2.1. *Suppose that w is a quasiconformal Jordan curve in the complex plane. Let ϕ be an analytic function defined in a neighborhood of w which maps w onto itself homeomorphically. Moreover, suppose that ϕ has no periodic points on w. Then, there is a homeomorphism h from w onto the unit circle so that*

$$h(\phi(z)) = e^{2\pi i\omega}h(z)$$

for all z and some irrational ω.

When w is a circle, this reduces to Yoccoz' theorem, see [**11**]. It is an interesting question, posed to the author by Peréz-Marco, whether the claim of Theorem 2.1 still holds when w is an arbitrary Jordan curve, or even just a continuum in the plane and $\phi(z)$ is understood as the action on prime ends.

Plan of the proof. Without loss of generality, the only critical points of ϕ are on w. Consider the Riemann mapping R from the unit disk onto W, one of the connected components of the complement of w. For z on the unit circle use the continuous extension of R to the boundary to define the map $f(z) = R^{-1} \circ \phi \circ R(z)$. The map f is a homeomorphism without periodic points, hence it must be of degree 1 with an irrational rotation number. Theorem 2.1 will follow from Proposition 1 once we show that the lifting of f, satisfies the cross-ratio inequality with respect to some cross-ratio module. This method of proof gives more than Theorem 2.1. For example, the lifting of f has bounded geometry properties from Lemma 1.2 and is conjugated to the rotation by a quasi-symmetric map provided that the rotation number is of constant type. The function f is easily seen to be real-analytic except at the preimages by R of the critical points of ϕ. There appears to be no easy way of controlling the behavior of f near these, let us call them exceptional, points. Hence, it is not clear whether the problem can be reduced to Yoccoz' theorem or even how to prove the cross-ratio inequality with respect to **Cr**. Choosing a convenient cross-ratio module, however, makes checking the cross-ratio inequality a relatively simple task.

2.1. Constructing the cross-ratio module

Let recall the notion of a *quadrilateral.* In the simplest setting, consider a Jordan curve w in the complex plane with four points on it, called the vertices of the quadrilateral. These points divide w into four arcs. Choose one of them as the *base* of the quadrilateral. Every such configuration defines a quadrilateral. The inner component of the complement of w can be mapped univalently onto the interior of a rectangle with vertices $(0, 1, 1 + ia, ia)$ in such a way that the base is mapped to the interval $(0, 1)$ be the continuous extension of the map. It turns out that a depends only on the quadrilateral as is called its *module.* It is useful to think of the family of admissible curves of a quadrilateral. These are curves contained inside the quadrilateral which join the base with the opposite side. The basic inequality is that if quadrilaterals Q_1 and Q_2 are related in such a way that the set of admissible curves for Q_1 is contained in the set of admissible curves for Q_2, then $\mathrm{mod}\ Q_1 \geq \mathrm{mod}\ Q_2$. A good exposition of the properties of conformal module can be found in [**7**], pages 19-28.

If w is the real line, it has a natural cyclic order of points which makes it positively oriented with respect to the upper half-plane. For example, we can talk of an arc $(1, -1)$ meaning the compactified real line minus $[-1, 1]$. In this situation if A, B, C, D are cyclically ordered points on w, define $Q(A, B, C, D)$ to be the quadrilateral with these vertices and the base equal to the arc (A, B).

Definition 2.1. *Consider points $A \leq B < C \leq D$ on the real line. Let*

$$\chi(A, B, C, D) = \frac{1}{\mathrm{mod}\ Q(A, B, C, D)} .$$

If $A = B$ or $C = D$, we set $\chi(A, B, C, D) = 0$. Also, define

$$\chi(A, B, C, D) := \chi(D, C, B, A)$$

if the foregoing definition makes sense for the reversed sequence.

We will now check that χ defines a cross-ratio module in the sense of Definition 1.2.

For a quadrilateral from a certain class, we can associate its module with the better known modulus of a ring domain. Pick cyclically ordered points A, B, C, D on the real line and consider a Jordan arc v inside the closed upper half-plane with endpoints at A and D. Then we have a quadrilateral Q with vertices A, B, C, D, base (A, B) and the remaining sides (B, C), (C, D) and v. If Q' is the image of Q by the reflection about the real line, then the union of the interiors of Q', Q together with arcs (A, B) and (C, D) forms a ring domain $R(Q)$. We will say that this ring domain is *associated* with Q. Then

$$\mathrm{mod}\ R(Q) = \frac{\pi}{\mathrm{mod}\ Q} .$$

So,

$$\chi(A, B, C, D) = \pi\ \mathrm{mod}\ R(Q(A, B, C, D)) .$$

We are ready to see that χ is a cross-ratio module. If $\mathbf{Cr}(A,B,C,D) \geq 1/4$, then both $|A-B|$ and $|C-D|$ are at least $1/5$ of $|B-C|$ and a round ring with large modulus can be fit inside $R(Q(A,B,C,D))$. The second condition of Definition 1.2 follows from Teichmüller's module theorem, see [**7**], page 56. This theorem says that if a ring domain R separates points w_1 and w_2 from ∞ and z, then

$$\text{mod } R \leq \psi(\frac{|z-w_1|}{|w_1-w_2|})$$

where ψ is a universal function whose limit at 0 is 0. Hence, $\chi(A,B,C,D) < \epsilon$ provided that $|A-B|/|C-B| < \delta_1(\epsilon)$ or $|C-D|/|C-B| < \delta_1(\epsilon)$ and this last condition is implied when $\mathbf{Cr}(A,B,C,D)$ is sufficiently small. So, χ indeed is a cross-ratio module.

2.2. The cross-ratio inequality

It is left to us to verify the cross-ratio inequality for the lifting of f and with respect to χ. Let us begin with a technical lemma.

Lemma 2.1. *Suppose that points A,B,C,D are cyclically ordered on the real line. For every $k>0$, if $|D-A| \leq k/2$, then there is a Jordan arc v with endpoints D and A so that the quadrilateral Q' with sides (A,B), (B,C) and (C,D) and the fourth side v has interior disjoint from $V_k = \{z \colon \Im z \geq k\}$ and satisfies*

$$\text{mod } Q' \leq \frac{\text{mod } Q(A,B,C,D)}{1-|D-A|/k} .$$

Proof. Since the situation is clearly translation-invariant, we can assume without loss of generality that $A=-D$ and $D>0$. Consider the map $M'(z) = \frac{2}{D-A}z$ from the upper half plane onto itself. M' sends A and D to -1 and 1, respectively, while $M'(V_k) = V_{\frac{2k}{D-A}}$. Next, consider the Möbius map M from the upper half-plane onto the unit disk which fixes -1 and 1, sends 0 to $-i$, ∞ to i and i to 0. One checks directly that

$$\Im M(\frac{2ki}{D-A}) > 1 - \frac{D-A}{k} .$$

Let $\Gamma = M \circ M'$. Then $\Gamma(V_k) \subset \{z \colon \Im z > 1 - \frac{D-A}{k}\}$.

Now choose $\Gamma(w)$ as the arc of the ellipse which joins -1 and 1 and avoids $\Gamma(V_k)$. The smaller semi-axis of this ellipse can be taken to be $1 - \frac{D-A}{k}$. This defines the quadrilateral $\Gamma(Q')$. To estimate its module, apply the homeomorphism H which is the identity in the lower half-plane and acts by

$$H(x+iy) = x + i\frac{y}{1-\frac{D-A}{k}}$$

in the upper half-plane. Obviously, H is

$$\frac{1}{1-\frac{D-A}{k}} - \text{quasiconformal},$$

at least 2-quasiconformal because of our hypothesis $|D-A| \leq k/2$. Hence,

$$\text{mod } Q' = \text{mod } \Gamma(Q') \leq \frac{\text{mod } H(\Gamma(Q'))}{1-\frac{D-A}{k}}$$

but $H(\Gamma(Q')) = \Gamma(Q)$. □

The cross-ratio inequality - the set up and the easy case. Choose a lift g of f. We have to check the condition of Definition 1.3. Let us consider the allowable configuration of quadruples of points (a_i, b_i, c_i, d_i). For any $\eta > 0$ and without loss of generality, all distances $d_i - a_i$ can be assumed less than η. This is because all values i for which this is violated are no more than $1/\eta$ in number and each contributes a bounded factor to the product in the cross-ratio inequality by continuity.

If η is chosen small, each interval (a_i, d_i) contains at most one exceptional point of g. Consider first the position without exceptional points. Consider $Q' := Q(g(a_i), g(b_i), g(c_i), g(d_i))$. The map G is holomorphic in some strip above the real axis and its range covers the strip $U_k := \{z\colon \Im z < k\}$ for some $k > 0$. By Lemma 2.1, if $\eta < k/2$, then inside $Q' \cap U_k$ we can find a quadrilateral Q'' with the same base two other sides and with

$$\text{mod } Q'' \leq \frac{\text{mod } Q'}{1 - C|g_i(a) - g_i(d)|} \tag{9}$$

where $C := k^{-1}$. Then Q'' can be mapped by an inverse branch of g to some quadrilatral which is contained in $Q(a_i, b_i, c_i, d_i)$. Hence,

$$\text{mod } Q(a_i, b_i, c_i, d_i) \leq \text{mod } Q'' \leq$$

$$\leq (1 - C|g(a_i) - g(d_i)|)^{-1} \text{mod } Q(g(a_i), g(b_i), g(c_i), g(d_i))$$

and,

$$\frac{\chi(g(a_i), g(b_i), g(c_i), g(d_i))}{\chi(a_i, b_i, c_i, d_i)} \leq \frac{1}{1 - C|g(d_i) - g(a_i)|}.$$

Since intervals $g(a_i, d_i)$ are disjoint modulo 1, their joint length is bounded by 1 and the product of contributions of this type for all i is also bounded.

The handling of exceptional points. Now suppose that (a_i, d_i) contains an exceptional point of g. We use, for the only but crucial time in this proof, the assumption that the initial curve w in the statement of Theorem 2.1 is quasiconformal. This means that the Riemann map used to define f extends globally as a quasiconformal homeomorphism. So, g extends to a strip on both sides of the real line in the form $G = H^{-1} \circ \phi \circ H$ where H is a K-quasiconformal covering map given as the composition of the exp with the global quasiconformal extension of the Riemann map R. Suppose that a symmetric strip U'_k of width k is covered by the range of G. As in the previous case, we first consider $Q' := Q(g(a_i), g(b_i), g(c_i), g(d_i))$ and then choose a smaller Q'' whose interior is contained in the upper half of U'_k and for which estimate (9) holds. Then construct the associated ring domain $R(Q'')$ which now fits inside U'_k. Also,

$$\begin{aligned} \text{mod } R(Q'') &\geq \frac{\pi(1 - C|g(a_i) - g(d_i)|)}{\text{mod } Q'} \\ &= \frac{\pi(1 - C\eta)}{\chi(g(a_i), g(b_i), g(c_i), g(d_i))} \end{aligned} \tag{10}$$

with $C := k^{-1}$.

Let us consider the preimage of $R(Q'')$ by G. Let γ denote the exceptional point which is contained in (a_i, d_i). Consider first the case when γ is in $[b_i, c_i]$. Then $R' := G^{-1}(R(Q''))$ is a ring domain and G restricted to R' is a K^2-quasiconformal covering of $R(Q'')$ of degree equal to the order of the critical point of ϕ at $H(\gamma)$. If this order is ℓ, then

$$\text{mod } R' \geq \frac{\text{mod } R(Q'')}{\ell K^2}.$$

If γ is not in $[b_i, c_i]$, let us assume, only to fix attention, that it belongs to (a_i, b_i). In that case we can find a Jordan curve t which passes through $g(\gamma)$ and splits $R(Q'')$ into two nesting annuli, say R_1 and R_2, so that

$$\text{mod } R_1 + \text{mod } R_2 = \text{mod } R(Q'') .$$

Suppose that R_1 is the outer layer, i.e. it surrounds $g(\gamma)$. If mod $R_1 \geq$ mod R_2, we can take as R' the preimage of R_1 by G. As in the previous case, we argue that

$$\text{mod } R' \geq \frac{\text{mod } R(Q'')}{2\ell K^2}.$$

If, on the other hand, mod $R_2 >$ mod R_1, we can take as R' the preimage of R_2 by the inverse branch of G which agrees with g^{-1} on the real line. In this case,

$$\text{mod } R' \geq \frac{\text{mod } R(Q'')}{2K^2}.$$

Summarizing, in every case we get a ring domain R' which is contained in $R(Q(a_i, b_i, c_i, d_i))$ and has modulus at least

$$\frac{\text{mod } R(Q'')}{2\ell K^2}.$$

We estimate

$$\chi(a_i, b_i, c_i, d_i) = \pi^{-1} \text{mod } R(Q(a_i, b_i, c_i, d_i)) \geq \pi^{-1} \text{mod } R' \geq$$
$$\geq \frac{\text{mod } R(Q'')}{2\pi\ell K^2} \geq \frac{\chi(g(a_i), g(b_i), g(c_i), g(d_i))}{2\ell K^2(1 - C\eta)} .$$

The last inequality follows from estimate (10). The cross-ratio inequality follows, and thus Theorem 2.1.

Comments. We have seen that a lot of the work done in this section was independent of the hypothesis about the quasiconformality of w. The construction of χ and the proof of the cross-ratio inequality away from exceptional points would work for any Jordan curve. In fact, they would only require ϕ to be defined on one side of w. It is then not surprising that we had to use more in the case when an exceptional point is near.

On the other hand, to assume that the curve is quasiconformal is clearly too strong. There should be a local condition at the critical points of ϕ which makes the proof work.

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